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An identity involving the least common multiple of binomial coefficients and its application

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Abstract

In this paper, we prove the identity

$$\mathrm{lcm}\left\{\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right\} = \frac{\mathrm{lcm}(1, 2, \dots, k, k+1)}{k+1} \qquad (\forall k \in \mathbb{N}).$$

As an application, we give an easily proof of the well-known nontrivial lower bound $\mathrm{lcm}(1, 2, \dots, k) \geq 2^{k-1}$ ($\forall k \geq 1$).

MSC: 11A05.
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1 Introduction and Results

Many results concerning the least common multiple of a sequence of integers are known. The most famous is nothing else than an equivalent of the prime number theorem; it states that $\log \mathrm{lcm}(1, 2, \dots, n) \sim n$ as n tends to infinity (see, e.g., [4]). Effective bounds for $\mathrm{lcm}(1, 2, \dots, n)$ are also given by several authors. Among others, Nair [7] discovered a nice new proof for the well-known estimate $\mathrm{lcm}(1, 2, \dots, n) \geq 2^{n-1}$ ($\forall n \geq 1$). Actually, Nair's method simply exploits the integral $\int_0^1 x^n(1-x)^n dx$. Further, Hanson [3] already obtained the upper bound $\mathrm{lcm}(1, 2, \dots, n) \leq 3^n$ ($\forall n \geq 1$).

Recently, many related questions and many generalizations of the above results have been studied by several authors. The interested reader is referred to [1], [2], and [5].

In this note, using Kummer's theorem on the p-adic valuation of binomial coefficients (see, e.g., [6]), we obtain an explicit formula for $\mathrm{lcm}\{\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\}$

in terms of the least common multiple of the first $k+1$ consecutive positive integers. Then, we show how the well-known nontrivial lower bound $\mathrm{lcm}(1, 2, \dots, n) \geq 2^{n-1}$ $(\forall n \geq 1)$ can be deduced very easily from that formula. Our main result is the following:

Theorem 1 *For any $k \in \mathbb{N}$, we have:*

$$\mathrm{lcm}\left\{\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right\} = \frac{\mathrm{lcm}(1, 2, \dots, k, k+1)}{k+1}.$$

First, let us recall the so-called Kummer's theorem:

Theorem (Kummer [6]) *Let n and k be natural numbers such that $n \geq k$ and let p be a prime number. Then the largest power of p dividing $\binom{n}{k}$ is given by the number of borrows required when subtracting k from n in the base p.*

Note that the last part of the theorem is also equivalently stated as the number of carries when adding k and $n-k$ in the base p.

As usually, if p is a prime number and $\ell \geq 1$ is an integer, we let $v_p(\ell)$ denote the normalized p-adic valuation of ℓ; that is, the exponent of the largest power of p dividing ℓ. We first prove the following proposition.

Proposition 2 *Let k be a natural number and p a prime number. Let $k = \sum_{i=0}^{N} c_i p^i$ be the p-base expansion of k, where $N \in \mathbb{N}$, $c_i \in \{0, 1, \dots, p-1\}$ (for $i = 0, 1, \dots, N$) and $c_N \neq 0$. Then we have:*

$$\max_{0 \leq \ell \leq k} v_p\left(\binom{k}{\ell}\right) = v_p\left(\binom{k}{p^N - 1}\right) = \begin{cases} 0 & \text{if } k = p^{N+1} - 1 \\ N - \min\{i \mid c_i \neq p-1\} & \text{otherwise.} \end{cases}$$

Proof. We distinguish the following two cases:
1st case. If $k = p^{N+1} - 1$:
In this case, we have $c_i = p-1$ for all $i \in \{0, 1, \dots, N\}$. So it is clear that in base p, the subtraction of any $\ell \in \{0, 1, \dots, k\}$ from k doesn't require any borrows. It follows from Kummer's theorem that $v_p\left(\binom{k}{\ell}\right) = 0$, $\forall \ell \in \{0, 1, \dots, k\}$. Hence

$$\max_{0 \leq \ell \leq k} v_p\left(\binom{k}{\ell}\right) = v_p\left(\binom{k}{p^N - 1}\right) = 0,$$

as required.

2^{nd} case. If $k \neq p^{N+1} - 1$:
In this case, at least one of the digits of k, in base p, is different from $p-1$. So we can define:

$$i_0 := \min\{i \mid c_i \neq p-1\}.$$

We have to show that for any $\ell \in \{0, 1, \dots, k\}$, we have $v_p(\binom{k}{\ell}) \leq N - i_0$, and that $v_p(\binom{k}{p^N - 1}) = N - i_0$.
Let $\ell \in \{0, 1, \dots, k\}$ be arbitrary. Since (by the definition of i_0) $c_0 = c_1 = \dots = c_{i_0 - 1} = p - 1$, during the process of subtraction of ℓ from k in base p, the first i_0 subtractions digit-by-digit don't require any borrows. So the number of borrows required in the subtraction of ℓ from k in base p is at most equal to $N - i_0$. According to Kummer's theorem, this implies that $v_p(\binom{k}{\ell}) \leq N - i_0$.
Now, consider the special case $\ell = p^N - 1 = \sum_{i=0}^{N-1} (p-1)p^i$. Since $c_0 = c_1 = \dots = c_{i_0-1} = p - 1$ and $c_{i_0} < p - 1$, during the process of subtraction of ℓ from k in base p, each of the subtractions digit-by-digit from the rank i_0 to the rank $N - 1$ requires a borrow. It follows from Kummer's theorem that $v_p(\binom{k}{p^N - 1}) = N - i_0$. This completes the proof of the proposition. ■

Now we are ready to prove our main result.
Proof of Theorem 1. The identity of Theorem 1 is satisfied for $k = 0$. For the following, suppose $k \geq 1$. Equivalently, we have to show that

$$v_p\left(\operatorname{lcm}\left\{\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right\}\right) = v_p\left(\frac{\operatorname{lcm}(1, 2, \dots, k, k+1)}{k+1}\right), \tag{1}$$

for any prime number p.
Let p be an arbitrary prime number and $k = \sum_{i=0}^{N} c_i p^i$ be the p-base expansion of k (where $N \in \mathbb{N}$, $c_i \in \{0, 1, \dots, p-1\}$ for $i = 0, 1, \dots, N$, and $c_N \neq 0$). By Proposition 2, we have

$$\begin{aligned} v_p\left(\operatorname{lcm}\left\{\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}\right\}\right) &= \max_{0 \leq \ell \leq k} v_p\left(\binom{k}{\ell}\right) \\ &= \begin{cases} 0 & \text{if } k = p^{N+1} - 1 \\ N - \min\{i \mid c_i \neq p-1\} & \text{otherwise.} \end{cases} \end{aligned} \tag{2}$$

Next, it is clear that $v_p(\operatorname{lcm}(1, 2, \dots, k, k+1))$ is equal to the exponent of the largest power of p not exceeding $k+1$. Since (according to the expansion of k in base p) the largest power of p not exceeding k is p^N, the largest power of p not exceeding $k+1$ is equal to p^{N+1} if $k+1 = p^{N+1}$ and equal to p^N if $k + 1 \neq p^{N+1}$. Hence, we have

$$v_p\left(\operatorname{lcm}(1, 2, \dots, k, k+1)\right) = \begin{cases} N+1 & \text{if } k = p^{N+1} - 1 \\ N & \text{otherwise.} \end{cases} \tag{3}$$

Further, it is easy to verify that

$$v_p(k+1) = \begin{cases} N+1 & \text{if } k = p^{N+1}-1 \\ \min\{i \mid c_i \neq p-1\} & \text{otherwise.} \end{cases} \tag{4}$$

By subtracting the relation (4) from the relation (3) and using an elementary property of the p-adic valuation, we obtain

$$v_p\left(\frac{\operatorname{lcm}(1,2,\dots,k,k+1)}{k+1}\right) = \begin{cases} 0 & \text{if } k = p^{N+1}-1 \\ N - \min\{i \mid c_i \neq p-1\} & \text{otherwise.} \end{cases} \tag{5}$$

The required equality (1) follows by comparing the two relations (2) and (5). ■

2 Application to prove a nontrivial lower bound for $\operatorname{lcm}(1,2,\dots,n)$

We now apply Theorem 1 to obtain a nontrivial lower bound for the numbers $\operatorname{lcm}(1,2,\dots,n)$ $(n \geq 1)$.

Corollary 3 *For all integer $n \geq 1$, we have:*

$$\operatorname{lcm}(1,2,\dots,n) \geq 2^{n-1}.$$

Proof. Let $n \geq 1$ be an integer. By applying Theorem 1 for $k = n-1$, we have:

$$\begin{aligned} \operatorname{lcm}(1,2,\dots,n) &= n \cdot \operatorname{lcm}\left\{\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1}\right\} \\ &\geq n \cdot \max_{0 \leq i \leq n-1} \binom{n-1}{i} \\ &\geq \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}, \end{aligned}$$

as required. The corollary is proved. ■

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